UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
Computer Engineering

ECE 204 *Numerical methods*

Approximating the derivative using least-squares best-fitting polynomials

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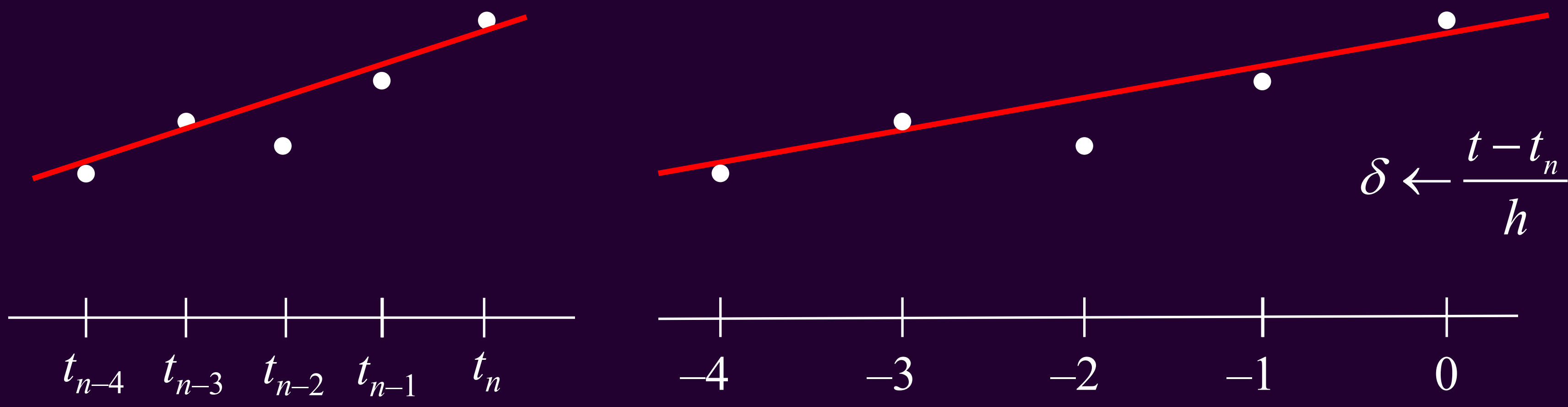
Introduction

- In this topic, we will
  - Discuss how to estimate the derivative of data by using the least-squares best-fitting polynomials
  - Estimating $y^{(1)}(t_n)$ or $y^{(2)}(t_n)$ where $t_k = t_0 + kh$
  - Describe the formula for both linear and quadratic polynomials
  - For the quadratic polynomial,
    we will also approximate the second derivative

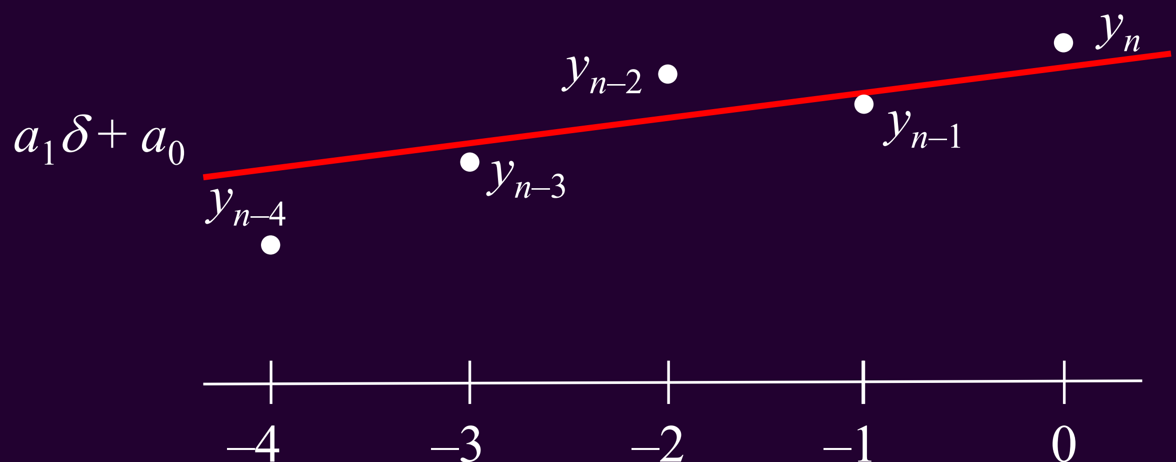
Approximating the derivative

- In the last lecture, we saw that
  if we mapped the last $N + 1$ times $t_{n-N}, \ldots, t_n$, to $-N, \ldots, 0$,
  we can easily find $a_1\delta + a_0$ on the right
  - In this case, $y(t_n + \delta h) \approx a_0 + a_1\delta$
  - The scaling, however, reduces the slope by h



Approximating the derivative

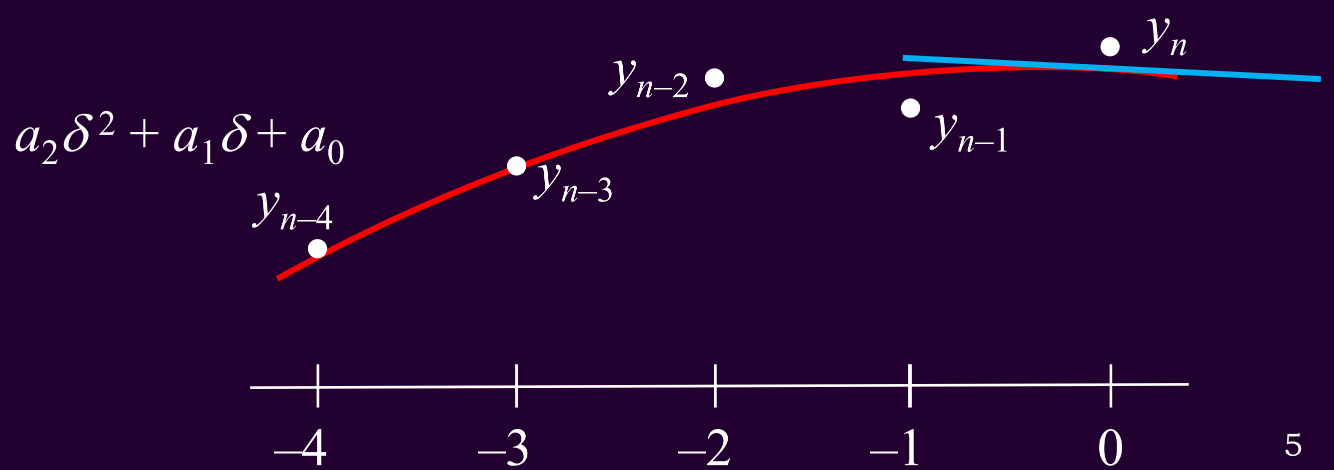
- Thus, for a least-squares best-fitting linear polynomial, the best estimate of the slope is $y^{(1)}(t_n) \approx \frac{a_1}{h}$
  - This is equal for all $t_{n-1} < t \le t_{n+1}$ under the assumption the original system has an approximately constant rate of change



Approximating the derivative

- Of the system is potentially accelerating or decelerating, we must use a least-squares best-fitting quadratic
  - If that quadratic polynomial is $a_2\delta^2 + a_1\delta + a_0$, the derivative is $2a_2\delta + a_1$
  - Thus, $y^{(1)}(t_n + \delta h) \approx \dfrac{2a_2\delta + a_1}{h}$

$$\delta \leftarrow \frac{t - t_n}{h}$$

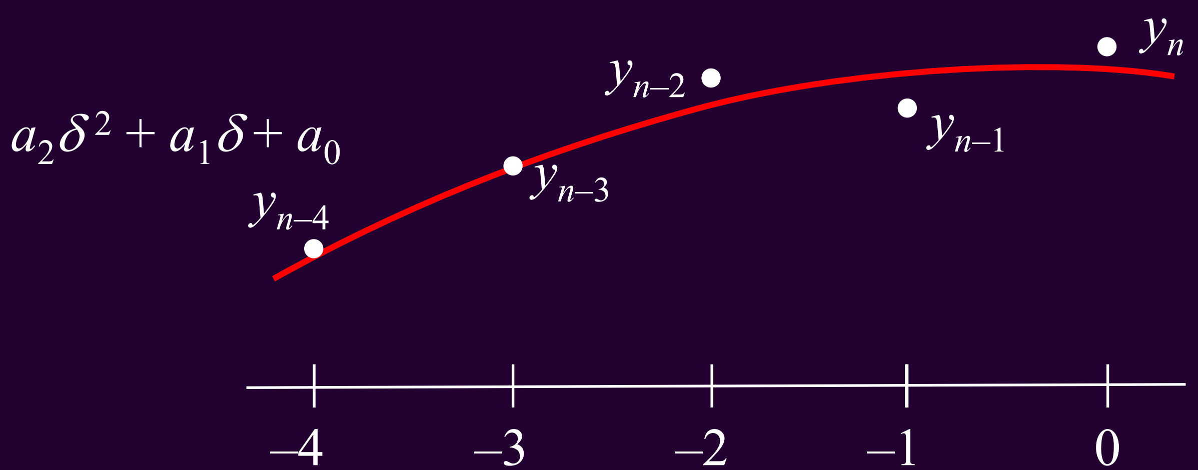


Approximating the second derivative

- Similarly, we can estimate the acceleration (concavity) with the formula

$$y^{(2)}(t_n) \approx \frac{2a_2}{h^2}$$

  – This formula accounts for scaling, as well



O(1) run time

This exemplifies an idea,
it is not required for this course.

- Everything in this class runs in O(1) time with O(n) memory:

```
class Estimate {
    public:
        Estimate( double y0, double delta_t );
        double operator()( double delta ) const;
        void next( double y );
        double diff() const;
private:
        std::size_t curr_;
        double      ys_[4];
        double      a1_, a0_, s_;
        double      delta_t_;
};

Estimate::Estimate( double y0, double delta_t ):
curr_{ 0 },
ys_{ y0, y0, y0, y0 },
a1_{ 0.0 },
a0_{ y0 },
sum_ { 4*y0 },
delta_t_{ delta_t } {
    // Empty
}
```

```
double Estimate::diff() const {
    return a1_/delta_t_;
}
```

Summary

- Following this topic, you now
  - Understand how to estimate the derivative and second derivative of least-squares polynomials
  - Are aware that because our formula involved scaling,
    we must divide the derivative by the step size,
    and the second derivative by the step size squared
  - Understand that if we already have the coefficients,
    we can find these estimates in O(1) time

References

[1] https://en.wikipedia.org/wiki/Least_squares

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

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